

A CHARACTERIZATION OF QUIVER ALGEBRAS BASED ON DOUBLE DERIVATIONS

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ABSTRACT. Let k a characteristic zero field. We give a characterization for the finite quiver k -algebras, based on double derivations. More precisely, we prove that if an associative and unitary k -algebra have a family of double derivations satisfying suitable conditions, then it is (canonically isomorphic to) a quiver algebra. This is the non-commutative version of a result of D. Wright.

INTRODUCTION

Let k be a characteristic zero field and A a k -algebra. We recall from [B], [C-E-G] and [G-S] that a double derivation of A is a derivation $D: A \rightarrow A \otimes A$, where A and $A \otimes A$ are considered as A -bimodules in the standard way. Suppose by a moment that A is a commutative ring. In [W] (see also [C]) it was proved that if there exist elements $x_1, \dots, x_n \in A$ and commuting derivations $D_1, \dots, D_n: A \rightarrow A$ such that

$$D_i(x_j) = \delta_{ij}, \text{ where } \delta \text{ is the symbol of Kronecker, for } 1 \leq i, j \leq n,$$

then A is the polynomial ring $R[x_1, \dots, x_n]$, where R is the ring of constants of A (for the definition see the next section).

In this note we give a similar characterization for the finite quiver algebras, but based on double derivations. Namely, we prove that if an algebra A have a family of double derivations satisfying suitable conditions, then A is (canonically isomorphic to) a quiver algebra. We obtain this result as a corollary of another one, which is the exact noncommutative version of the above mentioned result of [W]. Our method of proof consists in adapt to the noncommutative setting the one given in that paper.

1. PRELIMINARIES

In this note k denotes a characteristic 0 field, an algebra means an associative and unitary k -algebra and the unadorned tensor product is the tensor product over k . Given a family $D_1, \dots, D_n: A \rightarrow A \otimes A$ of double derivations (see the introduction), we say that an element $a \in A$ is a *constant* of A if $D_i(a) = 0$ for all i . It is immediate that the constants form a subring of A .

Let A be an algebra and let $s, t \in A$. It is easy to check that the tensor algebra $T_A(As \otimes tA)$ is isomorphic to the algebra with underlying vector space

$$A \oplus \bigoplus_{j \geq 0} As \otimes (tAs)^{\otimes j} \otimes tA$$

and multiplication

$$(a_1 \otimes \dots \otimes a_m)(a_{m+1} \otimes \dots \otimes a_n) = a_1 \otimes \dots \otimes a_m a_{m+1} \otimes \dots \otimes a_n.$$

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An isomorphism is the homogeneous map defined as the identity map in degree 0 and by

$$(a_1 s \otimes ta'_1) \otimes_A \cdots \otimes_A (a_n s \otimes ta'_n) \mapsto a_1 s \otimes \alpha_1 \otimes \cdots \otimes \alpha_{n-1} \otimes ta'_n,$$

where $\alpha_j = ta'_j a_{j+1} s$, in degree greater than 0. From now on we will use freely this identification.

Each double derivation D of A extends to a derivation

$$D: T_A(A \otimes A) \rightarrow T_A(A \otimes A).$$

via

$$D(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^n a_0 \otimes \cdots \otimes D(a_i) \otimes \cdots \otimes a_n.$$

We will say that D is *locally nilpotent* if for each $a \in A$ there exists $n \in \mathbb{N}$ (depending on a) such that $D^n(a) = 0$.

Proposition 1.1. *If A is an algebra and $D: A \rightarrow A \otimes A$ is a locally nilpotent double derivation, then the map*

$$\rho: A \rightarrow T_A(A \otimes A),$$

defined by

$$\rho(a) = a + D(a) + \frac{D^2(a)}{2!} + \frac{D^3(a)}{3!} + \frac{D^4(a)}{4!} + \cdots$$

is an injective morphism of algebras.

Proof. Straightforward. □

2. MAIN RESULT

Let A be an algebra, $X = \{x_1, \dots, x_n\}$ a set of elements of A and $B = A/\langle X \rangle$ the quotient algebra of A by the two sided ideal generated by X . Given $a \in A$, we let \bar{a} denote the class of a in B .

Theorem 2.1. *If there exist maps*

$$\begin{array}{ccc} X & \xrightarrow{s} & A \\ x_i & \longmapsto & s_i \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{t} & A \\ x_i & \longmapsto & t_i \end{array}$$

such that $s_i x_i t_i = x_i$ for all i and $\{s_1, \dots, s_n, t_1, \dots, t_n\}$ is a set of idempotent elements of A , and there exist locally nilpotent double derivations D_1, \dots, D_n of A satisfying

- (1) $D_i(s_j) = D_i(t_j) = 0$ for all i, j ,
- (2) $D_i(x_j) = \delta_{ij} s_j \otimes t_j$ for all i, j ,
- (3) *The diagram*

$$\begin{array}{ccc} A & \xrightarrow{D_i} & A \otimes A \\ \downarrow D_j & & \downarrow D_j \otimes A \\ A \otimes A & \xrightarrow{A \otimes D_i} & A \otimes A \otimes A \end{array}$$

commutes for all $i \neq j$,

then there is an isomorphism $\bar{\rho}: A \rightarrow T_B(M)$, where

$$M = \bigoplus_{1 \leq i \leq n} B \bar{s}_i \otimes \bar{t}_i B,$$

such that $\bar{\rho}(x_i) = \bar{s}_i \otimes \bar{t}_i$ and $\bar{\rho}^{-1}B$ is the ring of constants of A .

Lemma 2.2. *Let A be an algebra and D a locally nilpotent double derivation of A . Assume that for some $x \in A$ one has $D(x) = s \otimes t$, where $s, t \in A$ are idempotent elements such that $sxt = x$. Assume also that $D(s) = D(t) = 0$. Then, the map*

$$\bar{\rho} := A \xrightarrow{\rho} T_A(A \otimes A) \xrightarrow{\pi} T_B(B\bar{s} \otimes \bar{t}B),$$

where $B = A/\langle x \rangle$ and π is the canonical surjection, is an isomorphism of algebras. Moreover, this map identifies B with the ring of constants of A .

Proof. Suppose that $\bar{\rho}(a) = 0$ for some $a \neq 0$. We assert that for all $n \in \mathbb{N}$ there exists $\sum a^{(1)} \otimes \cdots \otimes a^{(n)} \in A^{\otimes n}$ such that

$$a = \sum a^{(1)}xa^{(2)}x \cdots xa^{(n-1)}xa^{(n)}.$$

For $n = 1$ this is immediate and for $n = 2$ it follows easily from the fact that the class of a in B is zero. Assume that it is true for n . Then

$$\bar{\rho}(a) = 0 \Rightarrow \sum \pi D^{n-1}(a^{(1)}xa^{(2)} \cdots a^{(n-1)}xa^{(n)}) = \pi D^{n-1}(a) = 0.$$

Since D is a derivation, $\pi(x) = 0$ and $D(x) = x + s \otimes t$, this implies that

$$\sum \pi(a^{(1)}s \otimes ta^{(2)}s \otimes \cdots \otimes ta^{(n-1)}s \otimes ta^{(n)}) = 0.$$

Hence,

$$\sum a^{(1)}s \otimes ta^{(2)}s \otimes \cdots \otimes ta^{(n-1)}s \otimes ta^{(n)} \in \sum_{i=0}^n A^{\otimes i} \otimes Ax A \otimes A^{\otimes n-i},$$

and the assertion holds for $n + 1$. Using now that

$$(2.1) \quad \sum \rho(a^{(1)})\rho(x)\rho(a^{(2)})\rho(x) \cdots \rho(x)\rho(a^{(n-1)})\rho(x)\rho(a^{(n)}) = \rho(a) \neq 0,$$

$\rho(x) = x + s \otimes t$ and $x = sxt$, it is easy to see that

$$(2.2) \quad \sum \rho(a^{(1)})s \otimes t\rho(a^{(2)})s \otimes \cdots \otimes t\rho(a^{(n-1)})s \otimes t\rho(a^{(n)}) \neq 0.$$

In fact, by (2.1), $\rho(a)$ is the sum of the homogeneous terms, obtained by replacing in the left side of (2.2) some of the $(s \otimes t)$'s by $sxt = x$. Since, each one of these terms is the image of the left side of (2.2) by an appropriate linear map, if $\rho(a) \neq 0$, then (2.2) holds. From (2.2) it follows that the degree of $\rho(a)$ is greater or equal than $n - 1$. Since n is arbitrary this is impossible, and so $\bar{\rho}$ is an injective map.

Since $\bar{\rho}(x) = \bar{s} \otimes \bar{t}$, in order to prove that $\bar{\rho}$ is surjective will be sufficient to check that its image contains B . For $a \in A$, let n_a be the minimal $n \in \mathbb{N}$ such that $D^j(a) \in \ker \pi$ for each $j \geq n$. We are going to prove that $\bar{a} \in \bar{\rho}A$, by induction on n_a . If $n_a = 1$, then $\bar{\rho}(a) = \bar{a}$. Suppose that $n_a = n + 1$ and $\bar{c} \in \bar{\rho}A$, for all $c \in A$ with $n_c \leq n$. Write

$$D^n(a) = \sum a_{(1)} \otimes \cdots \otimes a_{(n+1)}.$$

(Note that this is the Sweedler notation for the n -fold comultiplication of a in a bialgebra [S], but here D is a derivation instead of an algebra map). Let

$$L = \{c \in T_A(A \otimes A) : \pi D^i(c) = 0 \text{ for all } i \geq 0\}.$$

It is easy to see that

$$D^n(a) - \sum a_{(1)}s \otimes ta_{(2)}s \otimes \cdots \otimes ta_{(n)}s \otimes ta_{(n+1)}$$

and

$$\sum D^n(a_{(1)}xa_{(2)} \cdots a_{(n)}xa_{(n+1)}) - \sum a_{(1)}s \otimes ta_{(2)}s \otimes \cdots \otimes ta_{(n)}s \otimes ta_{(n+1)}$$

belong to L . Hence,

$$\pi D^j(a - \sum a_{(1)}xa_{(2)} \cdots a_{(n)}xa_{(n+1)}) = 0 \quad \text{for all } j \geq n,$$

and so, by the inductive hypothesis,

$$\overline{a - \sum a_{(1)} x a_{(2)} \cdots a_{(n)} x a_{(n+1)}} \in \overline{\rho} A.$$

Since $\overline{a} = \overline{a - \sum a_{(1)} x a_{(2)} \cdots a_{(n)} x a_{(n+1)}}$ this complete the proof. \square

Proof of Theorem 2.1. For $n = 1$ this is Lemma 2.2. Suppose that $n > 1$ and the result is valid for $n - 1$. Again by Lemma 2.2, we can assume that

$$A = T_{B_n}(B_n s_n \otimes t_n B_n),$$

where $B_n = \{a \in A : D_n(a) = 0\}$ and $D_n(s_n \otimes t_n) = s_n \otimes t_n \in B_n \otimes B_n$. From (1) and (2) it follows that $s_i, t_i, x_i \in B_n$ for all $i < n$, and from (3), that $D_i B_n \subseteq B_n$ for all $i < n$. Thanks the inductive hypothesis we can assume that

$$B_n = T_B(M),$$

where B is the ring of constants of A for D_1, \dots, D_n and $M = \bigoplus_{j=1}^{n-1} B s_j \otimes t_j B$. Hence

$$A = T_{B_n}(B_n s_n \otimes t_n B_n) = T_B(M \oplus B s_n \otimes t_n B),$$

as desired. \square

Corollary 2.3. *Under the hypothesis of Theorem 2.1, if $\{s_1, \dots, s_n, t_1, \dots, t_n\}$ is a complete set of orthogonal idempotents of the ring of constants $\overline{\rho}^{-1} B$ of A and*

$$\overline{\rho}^{-1} B = \bigoplus_{e \in \{s_1, \dots, s_n, t_1, \dots, t_n\}} k e,$$

then A is the quiver algebra kQ , where Q has vertices $Q_0 = \{s_1, \dots, s_n, t_1, \dots, t_n\}$ and one arrow $s_i \rightarrow t_i$ for each i .

Proof. It is immediate. \square

The converse of Theorem 2.1 is also true. That is, if B is an algebra and $s_1, \dots, s_n, t_1, \dots, t_n$ are (non-necessarily different) idempotents of B , then the algebra $A = T_B(M)$, where $M = \bigoplus_{i=1}^n B s_i \otimes t_i B$, has double derivations D_1, \dots, D_n satisfying (1) – (3), and moreover B is the ring of constants of A . When A is the non-commutative polynomial ring $k\{x_1, \dots, x_n\}$ (case $s_1 = \dots = s_n = t_1 = \dots = t_n = 1$) these are the partial double derivations considered in [C-E-G][Subsection 2.4]. When is a general quiver algebra they are those considered in [B].

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